Discussion Paper #2000-10

On a Problem of Proving the Existence of an Equilibrium in a Large Economy without Free Disposal: A problem of a purely finitely additive measure arising from the Fatou's lemma in several dimensions

by

Akira YAMAZAKI

February, 2001

On a Problem of Proving the Existence of an Equilibrium in a Large Economy without Free Disposal: A problem of a purely finitely additive measure arising from the Fatou's lemma in several dimensions

Akira Yamazaki
Graduate Faculty of Economics
Hitotsubashi University

February, 2000

Abstract

It is well-known in the literature that the free disposability or the desirability of all the commodities is needed to ensure the existence of an equilibrium in a large economy with a continuum of economic agents. This is a stark contrast to the case of economies with a finite number of agents, where an equilibrium can be shown to exist without assuming the free disposability nor the desirability of the commodities in case that negative prices are allowed to coordinate demands and supplies. This difference originates in the fact that feasible allocations to individuals are bounded by the totally available resources in finite economies whereas in economies with an infinite number of agents what each individual can feasibly consume need not be bounded by the average of totally available resources.

The purpose of our paper, however, is to show that the assumption of the free disposability nor the desirability of all the commodities is not needed for the existence of an equilibrium even in a large economy with a continuum of economic agents provided that negative prices are allowed and that the preference distribution among the agents satisfy a mild requirement that "if there are unboundedly desirable commodities, they must be unanimously regarded as such by almost all members of the economy."

1 Introduction

It is well-known in the literature that the free disposability or the desirability of all the commodities is needed to ensure the existence of an equilibrium in a large economy with a continuum of economic agents. This is a stark contrast to the case of economies with a finite number of agents, where an equilibrium can be shown to exist without assuming the free disposability nor the desirability of the commodities in case that negative prices are allowed to coordinate demands and supplies. This difference originates in the fact that feasible allocations to individuals are bounded by the totally available resources in case of finite economies whereas in case of economies with an infinite number of agents what each individual can feasibly consume need not be bounded by the average of totally available resources.

The purpose of our paper, however, is to show that the assumption of the free disposability nor the desirability of all the commodities is not needed for the existence of an equilibrium even in a large economy with a continuum of economic agents provided that negative prices are allowed and that the preference distribution among the agents satisfy a mild requirement that "if there are unboundedly desirable commodities, they must be unanimously regarded as such by almost all members of the economy."

In the literature there have been two types of proofs showing the existence of an equilibrium in a large economy. One is by Aumann [2] and Schmeidler [16] while the other is by Hildenbrand [8]. In both of these types, they first show the existence of equilibria in an environment where individual consumption sets are effectively bounded. Then, by considering a sequence of equilibria in an economy with bounded consumption sets where bounds are allowed to increase indefinitely, an equilibrium for the original economy is shown to exist by appealing to strictly positive limit prices in case of Aumann [2] and Schmeidler [16], and to the Fatou's lemma in several dimensions in case of Hildenbrand [8].

The strictly positive prices at a limit eventually establish bounds for budget sets along a convergent subsequence, implying that "bounded partial equilibria" along the subsequence eventually become equilibria.

However, the very reason that equilibria (without free disposal of commodities) existed is due to the fact that no one wanted to discard any commodities as all of the commodities are perceived as (unboundedly) desirable. Thus, strictly speaking from a theoretical point of view, one cannot answer the question of whether the market price mechanism can coordinate demands and supplies when the disposal activity of commodities is costly, if one assumes the desirability of all the commodities. The point here is that one needs to establish that market prices can indeed coordinate market forces of demand and supply even if unwanted commodities cannot be discarded freely in a large economy.

Once the assumption that all the commodities are desirable is dropped, we are in the set-up of Hildenbrand [8]. In that framework a difficulty arises at a step where the Fatou's lemma in several dimensions is applied to a subsequence. This lemma is applied to a subsequence in order to obtain an equilibrium with free disposal. In this step one cannot hope to obtain an equilibrium with exact feasibility unless the result of the Fatou's lemma in several dimensions is strengthened by appealing to an economic context.

Before we give a statement of the main result of this paper, we would like to take time to explain in more detail the nature of a difficulty in providing an equilibrium existence result in a large economy.

2 A Problem of a Purely Finitely Additive Measure Arising from the Fatou's Lemma in Several Dimensions

2.1 Two typical types of existence proofs

Let us briefly explain some basic features of existing existence proofs in large economies with a measure space of economic population so that the nature of difficulties in providing an existence proof of an equilibrium without assuming the free disposability of commodities nor assuming all the commodities to be unboundedly desirable become clear.

We will consider a large *economy* $\mathcal{E} : (A, \mathcal{A}, \nu) \to \mathcal{P} \times \mathbb{R}^\ell$ with an atomless measure space of economic population given by $(A, \mathcal{A}, \nu)$ with $\nu(A) = 1$. $\mathcal{E}$ is a measurable map, and $\mathcal{E}(a) = (\succ_a, e(a))$ with $e(a) \geqq 0$, $0 < \int e d\nu < \infty$ and a preference relation $\succ_a$ on $X_a \equiv \mathbb{R}^\ell_+$ which is continuous, *ifefi* open in $X_a \times X_a$, and *negatively transitive*, *ifefi* $z \not\succ_a x$ if $z \not\succ_a y$ and $y \not\succ_a x$.[1] In an integral we will often omit the symbol $d\nu$ and write $\int f$ instead of $\int f d\nu$. A preference relation $\succ_a$ is *locally nonsatiated* if for any $x \in X_a$ and for any neighborhood U of x there is $z \in U$ such that $z \succ_a x$. A preference relation $\succ_a$ defined on $\mathbb{R}^\ell_+$ is said to be *monotone* if for any $x \in \mathbb{R}^\ell_+$ and $v \in \mathbb{R}^\ell_+ \setminus \{0\}$ one has $x + v \succ_a x$. Commodity j is *desirable* if for any $x \in \mathbb{R}^\ell_+$ and any $t > 0$ one has $x + tu_j \succ_a x$ where $u_j \in \mathbb{R}^\ell$ is the vector with 1 in the j-th place and 0 elsewhere. If all the commodities are desirable, then the preference relation is monotone.

An *allocation* $f : A \to \mathbb{R}^\ell$ is an integrable function such that $f(a) \in X_a$ a.e. in A. It is *feasible* if $\int f \leqq \int e$, and *exactly feasible* if $\int f = \int e$.

A price vector is a member p of $\mathbb{R}^\ell$ such that $p \neq 0$. Note that negative prices are allowed to coordinate demands and supplies of possibly undesirable commodities, and that some of the prices may be 0.

An *equilibrium* for $\mathcal{E}$ is a pair (p, f) consisting of a price vector $p \in \mathbb{R}^\ell \setminus \{0\}$, and an allocation $f : A \to \mathbb{R}^\ell$ such that

[1] Here we are assuming the simplest type of consumption sets as in [2] and [16] since a way to deal with the case of consumption sets or the commodity space that need not be convex are well understood. See [18] and [19].

1. $f(a) \in B(a,p)$ and $B(a,p) \cap \succ_a (f(a)) = \emptyset$ a.e. $a \in A$, where $B(a,p) \equiv \{z \in X_a \mid p{\cdot}z \leqq p{\cdot}e(a)\}$ and, for each $y \in X_a$, $\succ_a (y) \equiv \{z \in X_a \mid z \succ_a y\}$.
2. $\displaystyle\int f = \int e$

If, in the above definition, one replaces the second condition by $\displaystyle\int f \leqq \int e$, permitting a possibility of having an inequality $\displaystyle\int f \neq \int e$, then the pair (p,f) is called an *equilibrium* with *free disposal*.

Define, for each positive integer $k = 1, 2, \ldots$, $k$fi*bounded budget sets* by

$$B^k(a,p) \equiv \left\{z \in X_a \;\middle|\; p \cdot z \leqq p \cdot e(a)\right\} \cap \left\{x \in \mathbb{R}^\ell_+ \;\middle|\; x \leqq k\Big(1 + \sum_{i=1}^{\ell} e^i(a)\Big)u\right\} \tag{2.1}$$

where u is the vector $(1, \ldots, 1)$.

A pair (p,f) consisting of a price vector $p \in \mathbb{R}^\ell \setminus \{0\}$ and an allocation f is a $k$fi*bounded partial equilibrium* for $\mathcal{E}$ if an equilibrium is defined with respect to $B^k(a,p)$ instead of $B(a,p)$

There are two strands of existing fundamental results on the existence of equilibrium in the literature. One is by Aumann [2] and by Schmeidler [16], and the other by Hildenbrand [8].

THEOREM by Aumann and Schmeidler (Aumann [2] and Schmeidler [16]): *Given an economy* $\mathcal{E}$fi*assume*fia.e. $a \in A$fi$\succ_a$ *satisfies the desirability of all commodities*fi *Then*fi *an equilibrium* (p,f) *exists with* $p \in \mathbb{R}^\ell_{++}$fi

THEOREM by Hildenbrand (Hildenbrand [8]): *Given an economy* $\mathcal{E}$fi *an equilibrium* (p,f) *with free disposal exists where* $p \in \mathbb{R}^\ell_+ \setminus \{0\}$fi

In the following two subsections we like to note and comment on the basic features of the proofs of these theorems so that the nature of the problem at hand is well understood.

2.2 Basic features of existing existence proofs

In this subsection we first briefly describe main steps of each of the proofs by Schmeidler and Hildenbrand schematically, and then turn to those of the Fatou's lemma in several dimensions.

2.2.1 Main steps of the proof by Schmeidler

Step SH1: Given an economy $\mathcal{E}$, a k-bounded partial equilibrium (p_k, f_k) exists with $p_k \in \mathbb{R}^\ell_+ \setminus \{0\}$ for each integer $k > 1$.

Step S2: One takes a convergent subsequence $p_k \to p$ and shows that the *desirability of all commodities* implies $p \in \mathbb{R}^\ell_{++}$.

Step S3: The last step is to show that the sequence of k-bounded partial equilibria (p_k, f_k) eventually becomes an equilibrium along the subsequence.

2.2.2 Main steps of the proof by Hildenbrand

Step SH1: Given an economy $\mathcal{E}$, a k-bounded partial equilibrium (p_k, f_k) with free disposal exists with $p_k \in \mathbb{R}^\ell_+ \setminus \{0\}$ for each integer $k > 1$.

Step H2: Given a sequence of k-bounded partial equilibria with free disposal, (p_k, f_k), $p_k \in \mathbb{R}^\ell_+ \setminus \{0\}$, it is shown that the *Fatoufs Lemma in several dimensions* implies the existence of an integrable function $f : A \to \mathbb{R}^\ell$ such that f is a limit point of $f_k(a)$ a.e. $a \in A$ and $\int f \leqq \int e$.

Step H3: It is shown that a pair (p, f) with p, a limit point of the sequence $\{p_k\}$, is an equilibrium with free disposal.

2.2.3 Fatou's Lemma in Several Dimensions

In order to see why in the last step of the Hildenbrand's proof one gets an equilibrium without an exact feasibility, we like to take time to see the source of this inexactness of feasibility. It originates in the Fatou's lemma in several dimensions. The statement of the lemma is given below and the main steps of it's proof will follow.

[Fatou's Lemma in Several Dimensions] *Let* $f_k : (A, \mathcal{A}, \nu) \to \mathbb{R}^\ell_+$*fi* $k = 1, 2, \ldots$*fi be integrable and* $\lim_k \int f_k$ *existsfi Thenfithere exists an integrable function* $f : A \to \mathbb{R}^\ell_+$ *such that*

fifi $f(a)$ *is a limit point of the sequence* $\{f_k(a)\}$*fiafiefi in Afi*

fifi $\int f \leqq \lim_k \int f_k$*fi*

This lemma was first proved by Schmeidler [17]. To understand a mathematical difficulty involved in applying the lemma to the sequence of k-bounded partial equilibrium to obtain an equilibrium at "a limit" as in the proof in 2.2.2, we like to show next the main steps of the proof by Hildenbrand and Mertens [10].

Proof by Hildenbrand and Mertens:
Step 1: Define $\mu_k(E) = \int_E f_k d\nu$ for $E \in \mathcal{A}$ and each $k = 1, 2, \ldots$. Then, $\mu_k \in \mathrm{ba}^\ell$, the

ℓ-fold product of bounded additive measures on $(A, \mathcal{A})$. Since $\{\mu_k(A)\}_k$ is bounded, by the Theorem of Alaoglu $\{\mu_1, \mu_2, \dots\}$ is relative $\sigma^\ell(ba, L_\infty)$-compact. Thus, $\{\mu_k(A)\}_k$ has $\sigma^\ell(ba, L_\infty)$-accumulation point $\mu \in \mathrm{ba}^\ell$.

Step 2: By the Theorem of Yoshida-Hewitt μ can be decomposed into two parts in such a way that it can be written as

$$\begin{aligned} \mu = \mu_c + \mu_p, &\qquad \mu_c, \mu_p \geqq 0 \\ \mu_c \in \mathrm{ca}^\ell, &\qquad \mu_p \text{ is purely finitely additive,} \end{aligned} \tag{2.2}$$

where ca^ℓ is the ℓ-fold product of countably additive measures on $(A, \mathcal{A})$.

Take a Radon-Nikodym derivative g of μ_c with respect to ν. Then, one has

$$\int g = \mu_c(A) \leqq \mu(A) = \lim_k \int f_k.$$

It follows that

$$\int g \leqq \lim_k \int f_k = \int e.$$

However, at this stage one *cannot* say that $g(a)$ is a limit point of $\{f_k(a)\}$, a.e. in A. In order to achieve this one needs one more step.

Step 3: One can show that there are $\delta_k^i \geqq 0, i = 0, \dots, \ell$, with $\sum_{i=0}^\ell \delta_k^i = 1$, and $y_k^i, i = 0, \dots, \ell$, each in the set $\{f_k(a), f_{k+1}(a), \dots\}$ satisfying

$$\begin{aligned} g(a) &= \lim_n \sum_{i=0}^\ell \delta_{k_n}^i y_{k_n}^i \\ &= \sum_{i=0}^\ell \lim_n \delta_{k_n}^i y_{k_n}^i \geqq \sum_{\substack{i=0 \\ \delta^i > 0}}^\ell \delta^i \lim_n y_{k_n}^i \end{aligned}$$

where $\delta^i = \lim_n \delta_{k_n}^i$.

2.3 A problem of a purely finitely additive measure arising from the Fatou's lemma in several dimensions

A problem underlying in proving the existence of an equilibrium in a large economy is clearly understood by examining the steps of typical proofs provided in above subsections 2.2.1, 2.2.2, and 2.2.3. The first proof by Aumann [2] and a subsequent proof by Schmeidler [16] as illustrated by the steps in 2.2.1 relies on the assumption that all the commodities are (unboundedly) desirable. This assumption was essential in establishing Step S2 where it is shown that there is a subsequence of k-bounded partial equilibria whose prices converge to strictly positive ones. The strictly positive prices at the limit eventually establish bounds for budget sets along the convergent subsequence, implying that k-bounded partial equilibria along the subsequence eventually become equilibria.

However, the very reason that equilibria (without free disposal of commodities) existed is due to the fact that no one wanted to discard any commodities as all of the commodities are perceived as (unboundedly) desirable. Thus, strictly speaking from a theoretical point of view, one cannot answer the question of whether the market price mechanism can coordinate demands and supplies when the disposal activity of commodities is costly, if one assumes the desirability of all the commodities. The point here is that one needs to establish that market prices can indeed coordinate market forces of demand and supply even if unwanted commodities cannot be discarded freely in a large economy.

Once the assumption that all the commodities are desirable is dropped, we are in the set-up of Hildenbrand [8] except that his model is that of a production economy. In that framework one could establish the existence of a k-bounded partial equilibrium without free disposal because budget sets are bounded. A difficulty arises in the last step where he applies the Fatou's lemma in several dimensions. His method of proof is to apply the Fatou's lemma in several dimensions to a sequence of k-bounded partial equilibria to obtain at "a limit point" an equilibrium with free disposal. In this step one *cannot* hope to obtain an equilibrium with *exact feasibility* unless the result of the Fatou's lemma in several dimensions is strengthened. More precisely, in the statement of the lemma in the subsection 2.2.3, the inequality in the second condition needs to be strengthened to equality.

In following the steps of the proof by Hildenbrand and Mertens [10], one sees that there appear to be two sources of this inequality. One is in Step 2. A purely finitely additive part of the weak limit of the sequence of bounded measures, generated by the sequence of allocations associated with k-bounded partial equilibria, gives rise to this inequality. The second source is in Step 3 where one ignores the terms with coefficients that go to zero, which implies that the corresponding $y^i_{k_n}$'s could be unbounded.

However it may appear that these two sources are independent, they all boil down to a non-vanishing purely finitely additive part in Step 2 of the proof of the Fatou's lemma. A problem is that it would reflect circumstances where a sequence of groups of agents with declining weights down to null are assigned commodity vectors $f_k(a)$ which are unbounded along the sequence of k-bounded partial equilibria. This type of phenomenon will not arise when all the commodities are assumed to be (unboundedly) desirable since strictly positive prices bound budget sets of agents.

Now, suppose that all the commodities are *not unboundedly desirable.* If we interpret this statement to mean that there is "a uniform bound or a uniformly perceived bound" for desirable commodities, then the problem referred to in the preceding paragraph should not arise. Undesirable commodities will not pause a problem to this issue as consumption sets are bounded from below. Thus, on an intuitive ground, only possible problematic case is the one where there is a diversified perception concerning the bounds or unboundedness of desirability of commodities. We shall give a formal statement of this intuition in the next section.

3 Statement of the Main Theorem

In order to obtain an equilibrium existence result without free disposal we shall introduce an assumption which requires unanimous perception among economic agents as to which commodities, if they exist, are unboundedly desirable.

Let $J = \{1, \ldots, \ell\}$ be the set of indices of all commodities. Define a subset $J^+_{\succ_a}$ of J consiting of "unboundedly desirable" commodities for agent a, that is, for each $a \in A$

$$J^+_{\succ_a} = \{j \in J \mid (\forall x \in X_a)(\exists t > 0) x + tu_j \succ_a x\}.$$

Commodity j is an unboundedly desirable commodity for agent a, *iff* $j \in J^+_{\succ_a}$, if, regardless of how much the agent already consumes the amount of that commodity, there always is a further increase of that commodity consumption which will be preferred by the agent.

If the preference relation is monotone, then all the commodities are unboundedly desirable. A desirable commodity is unboundedly desirable but not *vice versa*. In this paper we do not require monotonicity of preferences. In fact, it is not necessary that even one desirable commodity exists. Instead, what we require is that if it happens that a commodity is unboundedly desirable for a non-null set of agents, then this perception must be unanimously held by almost all agents.

Assumption [Unanimous Perception of Unboundedly Desirable Commodities (UPUDC)]: There exist a subset J^+ of J, possibly empty, and a positive number $\kappa > \sum_{i=1}^{\ell} \int e^i$ such that a.e. $a \in A$

1. $J^+_{\succ_a} = J^+$, and
2. $(\forall x \in X_a)\ x \not\succ_a x^\kappa(J \setminus J^+)$ where, for any $x \in \mathbb{R}^\ell_+$, $x^\kappa(J \setminus J^+)$ is defined by $\big(x^\kappa(J \setminus J^+)\big)^i = \min\{x^i, \kappa\}$ for $i \in J \setminus J^+$, and $\big(x^\kappa(J \setminus J^+)\big)^i = x^i$ for $i \in J^+$.

The above assumption of Unanimous Perception of Unboundedly Desirable Commodities (UPUDC) says that almost every agent in the economy unanimously agrees on which commodities, if any, are unboundedly desirable. The assumption is stated in two parts to express that either almost everyone wants a commodity unboundedly or there is a unanimously perceived bound as to how much each agent wants that commodity. Mathematically speaking, the second requirement can be weakened. That is, instead of requiring the unanimous perception of a bound κ of the commodities that are not unboundedly desirable, the bound can depend on each agent a as long as $\kappa(a)$ as a function from A into $\mathbb{R}^\ell_+$ is integrable. Of course, a part or the whole of the commodities could be "bads" or undesirable without violating the above assumption. It should go without saying that if almost everyone regards all the commodities are unboundedly desirable[2], then the assumption is automatically satisfied.

We now give a statement of our main theorem.

Theorem 1 *Given an economy* $\mathcal{E}$*fian equilibrium* (p, f) *exists with* $p \in \mathbb{R}^\ell \setminus \{0\}$ *provided that the preference distribution of* $\mathcal{E}$ *satisfies the assumption of Unanimous Perception of Unboundedly Desirable Commoditiesfi*

Our main theorem confirms our intuition that market prices can indeed coordinate market forces of supply and demand even if unwanted commodities cannot be discarded freely in a large economy if everyone agrees on as to which commodities are desirable without any bounds. A mathematical difficulty of a purely finitely additive measure arising from the Fatou's lemma is avoided because the unanimous perception of agents as to which commodities are unboundedly desirable induces either strictly positive prices for unboundedly desirable commodities that in turn limit the consumption of those commodities by agents, or agents themselves want not to consume commodities without limits.

4 Proof

Define

$$\begin{aligned}
K(a) &= \left\{ x \in \mathbb{R}^\ell_+ \,\middle|\, x \leqq k \left(1 + \sum_{i=1}^{\ell} e^i(a) \right) u \right\}, \\
K &= \left\{ x \in \mathbb{R}^\ell_+ \,\middle|\, x \leqq k \left(1 + \sum_{i=1}^{\ell} \int e^i \right) u \right\}, \\
P &= \left\{ p \in \mathbb{R}^\ell \,\middle|\, \sum_{i=1}^{\ell} |p^i| = 1 \right\}, \\
D^k_<(a, p) &= \{ x \in B(a, p) \mid (\forall z \in B_<(a, p))\, z \not\succ_a x \} \cap K(a).
\end{aligned} \tag{4.3}$$

The first step is to follow the standard proofs as in [2], [8], [16], and show the existence of an (quasi-)equilibrium in k-bounded economies. The only difference from these proofs is that negative prices are allowed so that prices can coordinate to achieve exact equality between demands and supplies even if some of the commodities are unwanted, *ifæfi*"bads", as in the case of the proofs of existence of an equilibrium (without free disposal nor monotonicity) in economies with a finite number of agents (see [12], [13],[7], [3], [15]).

For this purpose we shall define the correspondences $\pi : K \to P$, $\varphi : P \to K$, and $\Psi : P \times K \to P \times K$ by

[2]This would be the case if it is assumed that preferences are monotone.

$$
\begin{aligned}
\pi(x) &= \left\{ p \in P \;\middle|\; (\forall q \in P) p \cdot \left(x - \int e \right) \geqq q \cdot \left(x - \int e \right) \right\}, \\
\varphi(p) &= \left\{ x \in \mathbb{R}^{\ell}_{+} \;\middle|\; (\exists \text{ an integrable function } f : A \to \mathbb{R}^{\ell}_{+}) x = \int f \right. \\
&\qquad\qquad\qquad\qquad \left. \text{and } f(a) \in D^k_{<}(a,p) \text{ a.e. } a \in A \right\}, \\
\Psi(p,x) &= \pi(x) \times \varphi(p).
\end{aligned}
\tag{4.4}
$$

The correspondence Ψ is well defined and satisfies the conditions of Kakutani's fixed point theorem (see, e.g. [2] pp.8-10, [16] pp.581-582, or [18] pp.550-551). So, let $(p,x) \in \Psi(p,x)$. Since $x \in \varphi(p)$, there is an allocation $f : A \to \mathbb{R}^{\ell}_{+}$ such that $x = \int f$, and $f(a) \in D^k_{<}(a,p)$ a.e. $a \in A$. We shall show that the allocation f is exactly feasible. Suppose we had $\int f - \int e \neq 0$. Since $x = \int f \in \varphi(p)$, it follows that a.e. $a \in A$, $p \cdot f(a) \leqq p \cdot e(a)$. Thus, we have $p \cdot \left(\int f - \int e \right) \leqq 0$. On the other hand, since $p \in \pi(\int f)$, we would have

$$
p \cdot \left(\int f - \int e \right) \geqq \frac{1}{\sum_{i=1}^{\ell} \left| \int f^i - \int e^i \right|} \left(\int f - \int e \right) \cdot \left(\int f - \int e \right) > 0,
$$

which is a contradiction. Therefore, f must be exactly feasible.

Thus, we have established that for each integer $k \geqq 1$ there exists (p_k, f_k) satisfying

1. $p_k \in P$,
2. $f_k(a) \in D^k_{<}(a, p_k)$ a.e. $a \in A$, and
3. $\int f_k = \int e$.

Since P is compact, there is a convergent subsequence of the sequence $\{p_k\}$. We can assume without loss of generality that the sequence itself is convergent so that $p_k \to p$. We shall prove that the sequence (p_k, f_k) eventually becomes an equilibrium.

First, let

$$
W_p = \{ a \in A \mid p \cdot e(a) > \inf p \cdot X_a \}.
$$

If $(\exists j \in J) p^j < 0$, then $W_p = A$. If not, $p \geqq 0$ and thus $\int e > 0$ implies $p \cdot \int e > 0$, which in turn implies $\nu(W_p) > 0$. Define, then, an integer b and the set B by

$$
\begin{aligned}
b &> \frac{1}{\nu(W_p)} \int \sum_{i=1}^{\ell} e^i, \text{ and} \\
B &= \left\{ x \in \mathbb{R}_+^{\ell} \mid x \leqq bu \right\}.
\end{aligned} \tag{4.5}
$$

Lemma 1 *There is a subset $U \subset W_p$ of strictly positive measure having the property that*

$$
f_k(a) \in B \textit{ for infinitely many } k, \textit{ afefi } a \in U. \tag{4.6}
$$

Proof If not, for a.e. $a \in W_p$, $(\exists k_0) k > k_0 \Rightarrow f_k(a) \notin B$. So, for such k, $(\exists j) f_k^j(a) > b$. Thus, $\sum_{i=1}^{\ell} f_k^i(a) > b$. It follows that $\liminf_k \sum_{i=1}^{\ell} f_k^i(a) \geqq b$ and this implies

$$
\int_{W_p} \liminf_k \sum_{i=1}^{\ell} f_k^i(a) \geqq b\nu(W_p).
$$

Then, by Fatou's lemma we obtain

$$
\begin{aligned}
\int_{W_p} \liminf_k \sum_{i=1}^{\ell} f_k^i(a) &\leqq \liminf_k \int_{W_p} \sum_{i=1}^{\ell} f_k^i(a) \\
&\leqq \liminf_k \int \sum_{i=1}^{\ell} f_k^i(a) \\
&= \liminf_k \int \sum_{i=1}^{\ell} e^i(a) = \int \sum_{i=1}^{\ell} e^i(a).
\end{aligned} \tag{4.7}
$$

Therefore, we must have

$$
b\nu(W_p) \leqq \int \sum_{i=1}^{\ell} e^i(a) < b\nu(W_p),
$$

a contradiction. This establishes the above claim. ∎

Next, we prove the following lemma.

Lemma 2 *For every $j \in J^+$fiwe have $p^j > 0$fi*

Proof By lemma 1, the compactness of B implies that, for a.e. $a \in U$, $\{f_k(a)\}$ has a limit point $y \in B$. Taking a subsequence if necessary, one can assume $f_k(a) \to y$. Then,

$$
p \cdot y = \lim_k p_k \cdot f_k(a) = \lim_k p_k \cdot e(a) = p \cdot e(a).
$$

Now, suppose we had $p^j \leqq 0$ for $j \in J^+$. Then, since $j \in J^+ = J^+_{\succ_a}$, a.e. $a \in A$, by the assumption of unanimous perception of unboundedly desirable commodities,

$$(\exists z \in \mathbb{R}^\ell_+) z^j > \kappa, z^i = y^i \text{ for } i \neq j, \text{ and } z \succ_a y.$$

For this z, we have $p \cdot z \leqq p \cdot y = p \cdot e(a)$. $a \in W_p$ implies that there is $z_0 \in X_a = \mathbb{R}^\ell_+$ such that $p \cdot z_0 < p \cdot e(a)$. So, define for $n = 1, 2, \ldots$

$$z_n = \frac{1}{n} z_0 + \frac{n-1}{n} z.$$

Then, $z_n \to z$, and for all n, $p \cdot z_n < p \cdot e(a)$. For each n let $k_1(n)$ be such that $p_k \cdot z_n < p_k \cdot e(a)$ for $k \geqq k_1(n)$. As we have $z \succ_a y$, there is an integer n_0 such that $z_n \succ_a y$ for all $n \geqq n_0$. Since $f_k(a) \to y$, for each $n \geqq n_0$, there is an integer $k_2(n)$ greater than $k_1(n)$ such that we have $z_n \succ_a f_k(a)$ for $k \geqq k_2(n)$. This contradicts the fact that $f_k(a) \in D^k_<(a, p_k)$ a.e. $a \in A$. Therefore, we must have $p^j > 0$ for every $j \in J^+$. ∎

It follows from lemma 2 that there is a positive integer k_0 such that for all $j \in J^+$ we have $p^j_k \geqq \delta$ for all $k > k_0$ for some $\delta > 0$.

Now, let $z \in B(a, p_k)$ and $j \in J^+$. Then, we have

$$\begin{aligned}
\delta z^j \leqq p^j_k z^j &\leqq \sum_{i \in J^+}^{\ell} p^i_k z^i \\
&\leqq p_k \cdot e(a) - \sum_{i \notin J^+}^{\ell} p^i_k z^i \\
&\leqq \sum_{i=1}^{\ell} |p^i_k| e^i(a) + \sum_{i \notin J^+}^{\ell} |p^i_k| z^i \\
&\leqq \sum_{i=1}^{\ell} e^i(a) + \sum_{i \notin J^+}^{\ell} z^i \\
&\leqq \sum_{i=1}^{\ell} e^i(a) + (\ell - \sharp J^+)\kappa.
\end{aligned} \tag{4.8}$$

Thus, one obtains

$$z^j \leqq \frac{1}{\delta} \left(\sum_{i=1}^{\ell} e^i(a) + (\ell - \sharp J^+)\kappa \right). \tag{4.9}$$

Define an integer k_1 by

$$k_1 = \max \left\{ k_0, \frac{\kappa}{1 + \sum_i e^i(a)}, \frac{\sum_i e^i(a) + (\ell - \sharp J^+)\kappa)}{\delta(1 + \sum_i e^i(a))} \right\}.$$

Then, for any $k > k_1$ when $z \in B(a, p_k)$, we have $z_j \leqq k(1 + \sum_i e^i(a))$ for each $j \in J^+$.

Thus, if we have $z \notin K(a)$, then it must be that $z^i > \kappa$ for some $i \notin J^+$. So, define the subset J^- of $J \setminus J^+$, and a "partially truncated" vector z^κ of z by

$$\begin{aligned} J^- &= \{i \in J \setminus J^+ \mid z^i > \kappa\}, \\ (z^\kappa)^i &= \begin{cases} z^i & \text{if} \quad i \in J \setminus J^- \\ \kappa & \text{if} \quad i \in J^- \end{cases}. \end{aligned} \tag{4.10}$$

Since we have $z^\kappa \in K(a)$ and $f_k(a) \in D^k_<(a, p_k)$, $z^\kappa \not\succ_a f_k(a)$ for any $k > k_1$. It follows from the assumption of the unanimous perception of unboundedly desirable commodities that we have $z \not\succ_a z^\kappa$. Hence, by the negative transitivity of preference $\succ_a$ that $z \not\succ_a f_k(a)$ for any $k > k_1$. This establishes that

$$f_k(a) \in D(a, p_k), \qquad \text{a.e. } a \in A \qquad \text{for } k > k_1$$

where $D(a, p_k) = \{x \in B(a, p_k) \mid (\forall z \in B(a, p_k))\, z \not\succ_a x\}$. Therefore, a pair $(p_k, f_k(a))$ is an equilibrium for each $k > k_1$.

References

[1] ARROW, KENNETH J., AND GERARD DEBREU: "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, 22(1954), 265–290.

[2] AUMANN, ROBERT J.: "Existence of Competitive Equilibria in Markets with a Continuum of Traders," *Econometrica*, 34(1966), 1–17.

[3] BERGSTROM, THEODORE C.: "How to discard 'Free Disposability'- as No Cost," *Journal of Mathematical Economics*, 3(1976), 131–134.

[4] DEBREU, GERARD: "New Concepts and Techniques for Equilibrium Analysis," *Infi ternational Economic Review*,3(1962), 257–273.

[5] DEBREU, GERARD: "Existence of Competitive Equilibrium," Chapter 15 in KENNETH J. ARROW *et alfi* eds.: *Handbook of Mathematical Economics*, Amsterdam:North-Holland Publishing Co., Vol. II(1982), 697–743.

[6] GALE, DAVID, AND ANDREU MAS-COLELL: "An Equilibrium Existence Theorem for a General Model without Ordered Preferences," *Journal of Mathematical Economics*, 2(1975), 9–15.

[7] HART, OLIVER D., AND HAROLD W. KUHN: "A Proof of Existence of Equilibrium without the Free Disposal Assumption," *Journal of Mathematical Economics*, 2(1975), 335–343.

[8] HILDENBRAND, WERNER: "Existence of Equilibria for Economies with Production and a Measure Space of Consumers," *Econometrica*, 38(1970), 608–623.

[9] HILDENBRAND, WERNER: *Core and Equilibria of a Large Economyfi* Princeton: Princeton University Press, 1974.

[10] HILDENBRAND, WERNER, AND JEAN-FRANÇOIS MERTENS: "On Fatou's Lemma in Several Dimensions," *Zfi Wahrscheinlichkeitstheorie verwfi Gebfi* 17(1971), 151–155.

[11] MAS-COLELL, ANDREU: "An Equilibrium Existence Theorem without Complete or Transitive Preferences," *Journal of Mathematical Economics*, 1(1974), 237–246.

[12] MCKENZIE, LIONEL W.: "On the Existence of General Equilibrium for a Competitive Market," *Econometrica*, 27(1959), 54–71.

[13] MCKENZIE, LIONEL W.: "The Classical Theorem on Existence of Competitive Equilibrium," *Econometrica*, 49(1981), 819–841.

[14] NIKAIDO, HUKUKANE: "On the Classical Multilateral Exchange Problem," *Metroeconomica*, 8(1956), 135–145.

[15] SHAFER, WAYNE J.: "Equilibrium in Economies without Ordered Preferences or Free Disposal," *Journal of Mathematical Economics*, 3(1976),135–137.

[16] SCHMEIDLER, DAVID: "Competitive Equilibria in Markets with a Continuum of Traders and Incomplete Preferences," *Econometrica*, 37(1969), 578–585.

[17] SCHMEIDLER, DAVID: "Fatou's Lemma in Several Dimensions," *Proceedings of the American Mathematical Society*, 24(1970), 300–306.

[18] YAMAZAKI, AKIRA: "An Equilibrium Existence Theorem without Convexity Assumptions," *Econometrica*, 46(1978), 541–555.

[19] YAMAZAKI, AKIRA: "Diversified Consumption Characteristics and Conditionally Dispersed Endowment Distribution:Regularizing Effect and Existence of Equilibria," *Econometrica*, 49(1981), 639–654.